

A THEORY OF 3-CONNECTED GRAPHS

BY

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1. INTRODUCTION

A graph G consists of a set $E(G)$ of edges and a (disjoint) set $V(G)$ of vertices, together with a relation of *incidence* which associates with each edge two vertices, not necessarily distinct, called its *ends*. An edge is a *loop* if its ends coincide and a *link* otherwise. We call G a *finite* graph if the sets $V(G)$ and $E(G)$ are both finite, and a *null* graph if they are both null.

In this paper we consider only finite graphs. We denote the numbers of vertices and edges of a graph G by $\alpha_0(G)$ and $\alpha_1(G)$ respectively. The *valency* $\text{val}(G, x)$ of a vertex x of G is the number of incident edges, loops being counted twice. We shall denote the number of members of an arbitrary finite set S by $\alpha(S)$.

A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and each edge of H has the same ends in H as in G . We then write $H \subseteq G$. Suppose H_i is any class of subgraphs of G . Then we define the *union* (*intersection*) of the graphs H_i as the subgraph H of G such that $V(H)$ is the union (intersection) of the sets $V(H_i)$ and $E(H)$ is the union (intersection) of the sets $E(H_i)$.

Let n be a non-negative integer. We say G is *n-separated* if it is the union of two subgraphs H and K with the following properties:

- (i) $E(H) \cap E(K) = \emptyset$,
- (ii) $\alpha(V(H) \cap V(K)) \leq n$,
- (iii) Each of the subgraphs H and K has a vertex not belonging to the other.

Under these conditions we call the pair $\{H, K\}$ an *n-separator* of G . A graph which is not n -separated is called $(n+1)$ -connected. The 1-connected graphs are usually called simply the "connected graphs".

We call G *simple* if it has no loop and if no two links have the same pair of ends. This paper is concerned with those graphs which are both simple and 3-connected. The central problem is that of determining the simple 3-connected graphs of $m+1$ edges when those of m edges are known.

Special interest attaches to those simple 3-connected graphs which are planar. The application of our general results to these is discussed in section 6.

2. ESSENTIAL EDGES

Let $\{H, K\}$ be an n -separator of a graph G . Let U be the set of all vertices of $V(H) \cap V(K)$ which are not incident with edges of H . Let H_1 be the subgraph of G formed from H by removing the vertices of U . Then $\{H_1, K\}$ is an $(n - \alpha(U))$ -separator of G . Now let W be the set of all vertices of $V(H_1) \cap V(K)$ not incident with edges of K , and let K_1 be the subgraph of G formed by removing from K the vertices of W . Then $\{H_1, K_1\}$ is an $(n - \alpha(U) - \alpha(W))$ -separator of G . We call it a *reduced form* of $\{H, K\}$. If $V(H) \cap V(K)$ includes vertices of G of zero valency this reduced form will differ slightly from that obtained by interchanging H and K in the above construction.

A separator $\{H, K\}$ of G will be called *proper* if each vertex of $V(H) \cap V(K)$ is incident both with an edge of H and an edge of K . Reduced forms of separators are always proper.

Now let A be an edge of a simple 3-connected graph G . We write $G'(A)$ for the graph obtained from G by deleting A , but not its incident vertices. Evidently $G'(A)$ is simple. We write $G''(A)$ for the graph obtained from $G'(A)$ by identifying the ends u and v of A in G . This means that we replace u and v by a single vertex w , require that every edge incident with u or v in $G'(A)$ is to be regarded as incident with w , and postulate that the remaining incidence relations are the same in $G''(A)$ as in $G'(A)$. The graph $G''(A)$ has no loops, but it may have one or more pairs of links with the same ends.

We call A an *essential* edge of G if neither $G'(A)$ nor $G''(A)$ is both simple and 3-connected.

(2.1) Suppose $\alpha_0(G) \geq 4$ and $G'(A)$ is not 3-connected. Then $G'(A)$ has a proper 2-separator $\{H, K\}$ with the following properties:

- (i) $\alpha(V(H) \cap V(K)) = 2$,
- (ii) A has one end (in G) in $V(H)$ but not $V(K)$ and the other in $V(K)$ but not $V(H)$. (Fig. 1.)

Proof. By hypothesis $G'(A)$ has a 2-separator $\{H, K\}$. We may suppose this reduced to a proper 2-separator. Let H_0 be the subgraph of G obtained by adjoining to H the edge A and its ends u and v . If u and v are both in $V(H)$ it follows that $\{H_0, K\}$ is a 2-separator of G , contrary to hypothesis. Similarly u and v are not both in $V(K)$. This establishes (ii).

If $\alpha(V(H) \cap V(K)) < 2$ it follows that either $\{H_0, K\}$ is a 2-separator of G or $V(K) - V(H)$ has only a single vertex. But the former alternative is contrary to hypothesis. Hence $\alpha(V(H) - V(K)) = 1$ and similarly $\alpha(V(K) - V(H)) = 1$. But then $\alpha_0(G) = \alpha(V(H) - V(K)) + \alpha(V(K) - V(H)) + \alpha(V(H) \cap V(K)) \leq 3$, which is contrary to hypothesis. This establishes (i). In Fig. 1 we denote the common vertices of H and K by x and y , and we put $u \in V(H)$.

(2.2) Suppose $\alpha_0(G) \geq 4$ and that $G''(A)$ is not both simple and 3-connected. Then $G'(A)$ is the union of two subgraphs L and M of G with the following properties:

- (i) $E(L) \cap E(M) = \emptyset$,
- (ii) L and M have just three common vertices t , u and v , where the ends of A in G are u and v ,
- (iii) Each of t , u and v is incident with an edge of L and an edge of M ,
- (iv) L (but not necessarily M) has a vertex other than t , u and v . (Fig. 2.)

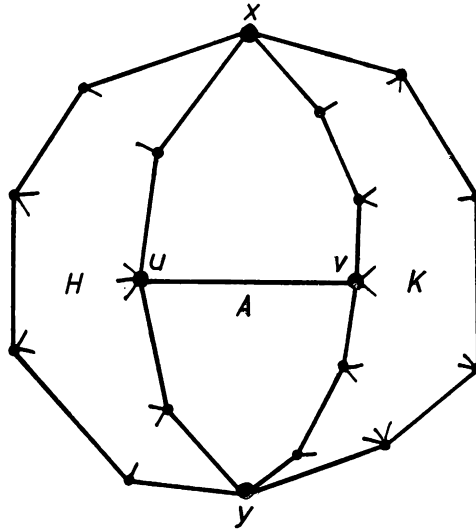


Fig. 1

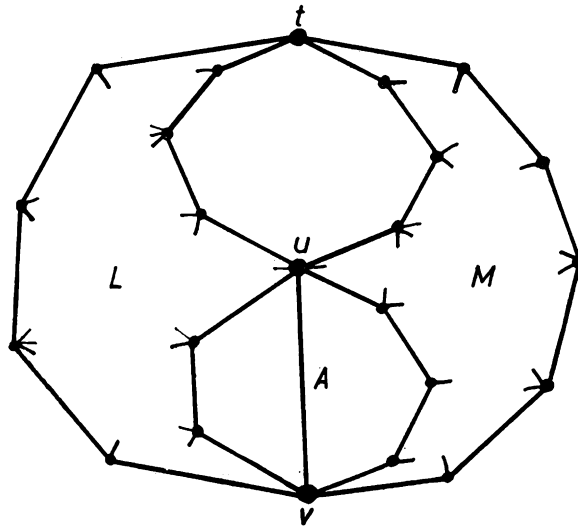


Fig. 2

Proof. We observe that $G''(A)$ is the union of two subgraphs P and Q of $G''(A)$ with the following properties:

- (1) $E(P) \cap E(Q) = \emptyset$,
- (2) P and Q have at most two common vertices,
- (3) P has a vertex not in $V(Q)$.

If $G''(A)$ is not 3-connected we take $\{P, Q\}$ to be a 2-separator of $G''(A)$. In the remaining case $G''(A)$ is not simple; it has two edges B and C with the same two ends. We then take Q to be the subgraph of $G''(A)$ defined by the edges B and C and their incident vertices. We define P by the equations $V(P) = V(G''(A))$ and $E(P) = E(G''(A)) - \{B, C\}$. Since $\alpha_0(G) \geq 4$ conditions (1)–(3) are then satisfied.

Let L be the subgraph of G defined by the edges of P and their incident vertices, and let M be defined similarly in terms of the edges of Q . The common vertices of L and M are the common vertices of P and Q except that if w is one of these it is to be replaced by one or both of u and v . By (3) L has a vertex not in $V(M)$.

We note that L and M satisfy conditions (i) and (iv). Assume they have not more than two common vertices. If M has a vertex not in $V(L)$ it follows that L is a member of a 2-separator of G , contrary to hypothesis. If M has no such vertex then Q has no vertex which is not in $V(P)$. This means that $E(Q)$ has just two members, these having the same ends in Q . Hence $\alpha_1(M) = 2$. Since G is simple it follows that M has at least three vertices, all of course in $V(L)$, which is contrary to our assumption.

From the above considerations it follows that (ii) is true. Then (iii) is true by the definition of L and M .

Finally $L \cup M = G'(A)$. For otherwise there is a vertex x of G having zero valency in $G'(A)$. This has also zero valency in G , by (ii) and (iii). Then $L \cup M$ is a member of a 2-separator of G , contrary to hypothesis.

We proceed to show that it is possible for all the edges of a simple 3-connected graph G to be essential. We shall use the following auxiliary theorem:

(2.3) *Let G be a 3-connected graph such that $\alpha_0(G) \geq 4$. Then $\text{val}(G, x) \geq 3$ for each $x \in V(G)$.*

Proof. Suppose $\text{val}(G, x) \leq 2$ for some $x \in V(G)$. Let H be the subgraph of G defined by x , its incident edges and their other ends. Let K be the subgraph defined by the edges not in $E(H)$ and the vertices other than x . Then $\alpha(V(H) \cap V(K)) \leq \text{val}(G, x) \leq 2$. Moreover $\alpha_0(G) \geq 4$ and $\alpha_0(H) \leq \text{val}(G, x) + 1 \leq 3$. Hence $\{H, K\}$ is a 2-separator of G , contrary to hypothesis.

A wheel of order n , where n is an integer ≥ 3 , is a graph W_n defined as follows:

- (i) $V(W_n) = \{a, b_0, b_1, \dots, b_{n-1}\}$,

- (ii) $E(W_n) = \{A_0, A_1, \dots, A_{n-1}, B_0, B_1, \dots, B_{n-1}\}$,
 (iii) The ends of A_i are a and b_i and the ends of B_i are b_i and b_{i+1} .

Here the suffices are residues mod n . The wheel W_5 is shown in Fig. 3.

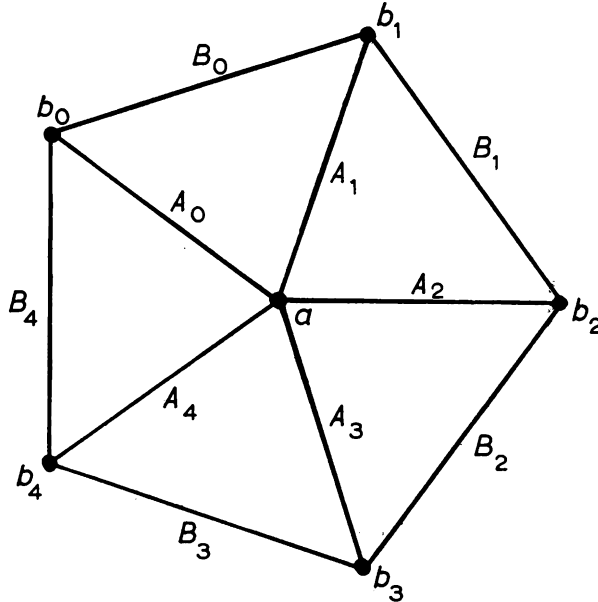


Fig. 3

(2.4) *Every wheel is simple and 3-connected.*

Proof. It is clear from the definition that W_n is simple. Let $\{H, K\}$ be any 2-separator of W_n . By the definition of an n -separator we can find $b_h \in V(H) - V(K)$ and $b_k \in V(K) - V(H)$, where $h+1 \neq k \neq h-1$. The common vertices of H and K include a member of the set $\{b_{h+1}, b_{h+2}, \dots, b_k\}$, a member of the set $\{b_{k+1}, b_{k+2}, \dots, b_h\}$, and the vertex a . Thus $\alpha(V(H) \cap V(K)) \geq 3$, contrary to the definition of a 2-separator. We deduce that W_n is 3-connected.

(2.5) *In a wheel every edge is essential.*

Proof. Put $G = W_n$. In $G'(A_i)$ and $G'(B_i)$ there is a vertex b_i of valency 2. Hence these graphs are not 3-connected, by (2.3). In $G''(A_i)$ and $G''(B_i)$ there are pairs of links, $\{B_i, A_{i+1}\}$ and $\{A_i, A_{i+1}\}$ respectively, with the same ends. Hence these graphs are not simple. The theorem follows.

3. TRIANGLES AND TRIADS

A *triangle* of a graph G is a set $\{A, B, C\}$ of three distinct edges with the following property: There exist vertices x, y and z such that the ends of A are x and y , the ends of B are y and z , and the ends of C are z and x .

A *triad* of G is a set $\{A, B, C\}$ of three distinct links of G with the following property: The three edges have a common end x which is

incident with no other edge of G . If G is simple the other ends of A , B and C are all distinct. We then call x the *centre* of the triad.

Let H be any subgraph of a graph G . A *vertex of attachment* of H in G is a vertex of H incident with an edge of G not in $E(H)$. We denote the set of all vertices of attachment of H in G by $W(G, H)$. It is clear that the following proposition holds:

(3.1) *If $\alpha(W(G, H)) < \alpha_0(H) < \alpha_0(G)$ then H is one member of an $\alpha(W(G, H))$ -separator of G .*

The other member is defined by the edges and vertices of G not belonging to H , together with the vertices of attachment of H .

Suppose H and K are subgraphs of G . A vertex of attachment of $H \cap K$ is incident with an edge of G not in H or with one not in K . Accordingly we have the following rule:

(3.2) *Any vertex of attachment of $H \cap K$ is either a vertex of attachment of H belonging to $V(K)$, or a vertex of attachment of K belonging to $V(H)$.*

We make frequent appeals to (3.2) in the rest of this section.

(3.3) *Let G be a simple 3-connected graph with at least 4 vertices. Let A be an essential edge of G . Then A belongs either to a triangle or to a triad of G .*

Proof. We combine the notations of (2.1) and (2.2) as depicted in Figs 1 and 2. Since x and y cannot both be vertices of attachment of L and M we may suppose $y \notin V(M)$, whence $y \neq t$. We may also adjust the notation, interchanging H and K if necessary, so that $t \notin V(H) - \{x\}$.

By (3.2) $H \cap M$ has no vertices of attachment other than x and u . For if t is such a vertex of attachment it is in $V(H)$ but not $V(H) - \{x\}$ and is thus x . But M has an edge B incident with u , by (2.2), and B must be in $E(H)$ by (2.1). Since G is simple and 3-connected it follows, by (3.1), that $H \cap M$ consists of the single edge B with its two ends x and u .

If $t \notin V(K) - \{x\}$, so that $t = x$, we find similarly that $K \cap M$ consists of a single edge C with its ends x and v . In this case $\{A, B, C\}$ is a triangle of G .

In the remaining case $t \in V(K) - \{x\}$. It follows that $x \notin V(L)$ since the edge B , incident with x , is in $E(M)$. Using (3.2) we find that $H \cap L$ has at most two vertices of attachment, u and y . But L has an edge D incident with u , and this must be in $E(H)$. Hence $H \cap L$ consists of the single edge D with its ends u and y . Since H is the union of $H \cap L$ and $H \cap M$ it follows that $\{A, B, D\}$ is a triad of G with centre u .

(3.4) *Let G be a simple 3-connected graph with at least 4 vertices. Let $\{A, B, C\}$ be a triangle of G such that A and B are essential. Then A belongs to a triad of G .*

Proof. Let the ends of A be u and v , and let those of B be v and w . In the case of A we use the notation of (2.1) as depicted in Fig. 1. We

note that w is incident with $B \in E(K)$ and $C \in E(H)$, and may therefore be identified with x .

By another application of (2.1) $G'(B)$ has a proper 2-separator $\{Q, R\}$ with the following properties:

- (i) $\alpha(V(Q) \cap V(R)) = 2$,
- (ii) B has one end v in $V(Q)$ but not $V(R)$ and the other end w in $V(R)$ but not $V(Q)$.

One common vertex of Q and R must be u . We denote the other by t . (See Fig. 4.)

Suppose A is the only edge of Q incident with u . Deleting A and u we obtain a subgraph Q_0 with no vertex of attachment other than t and v . But Q_0 includes each edge of Q incident with t . We deduce that Q_0 consists

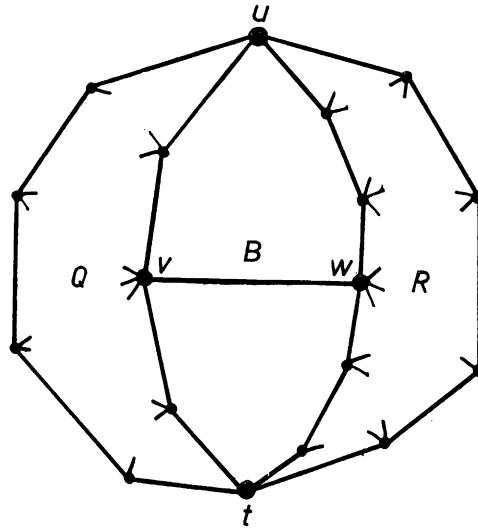


Fig. 4

of a single edge X and its two ends v and t . But then $\{A, B, X\}$ is a triad of G and the theorem is true. We may therefore assume that Q has an edge D incident with u and distinct from A .

Suppose $t \notin V(H)$, whence $t \neq y$. By (3.2) $H \cap Q$ has no vertex of attachment other than u and y . It therefore consists of the single edge D with its two ends u and y . Since $D \in E(Q)$ and $y \neq t$ this implies that $y \notin V(R)$.

It now follows from (3.2) that $H \cap R$ has no vertex of attachment other than x and u . We deduce that $H \cap R$ consists of the single edge C and its two ends x and u . Since $E(H)$ is the union of $E(H \cap R)$ and $E(H \cap Q)$ we deduce that $\{A, C, D\}$ is a triad of G with centre u .

In the remaining case $t \in V(H)$. By (3.2) the only possible vertices of attachment of $K \cap Q$ are y , v and t . But if $t \in W(G, K \cap Q)$ it belongs

to $V(K)$ as well as $V(H)$ and is therefore y . On the other hand all edges of G incident with v , except A and B , belong to both Q and K . There is at least one such edge by (2.3). We deduce that $K \cap Q$ has just one edge, E say, the ends of E being v and y . Hence $\{A, B, E\}$ is a triad of G with centre v .

(3.5) *Let G be a simple 3-connected graph with at least 4 vertices. Let $\{A, B, C\}$ be a triad of G such that A and B are essential. Then A belongs to a triangle of G .*

Proof. Let the centre of the triad be u and let the other ends of A, B and C be v, p and q respectively. In respect of A we use the notation of (2.2). (Fig. 2.)

Suppose M has no vertex other than t, u and v . Then by (iii) of (2.2) it must have one edge X incident with t and u , and another edge Y incident with t and v . But then $\{A, X, Y\}$ is a triangle and the theorem is true. We may now assume that M , like L , has a vertex distinct from t, u and v . Since u is the centre of the triad $\{A, B, C\}$ we may therefore assume without loss of generality that $B \in E(L)$ and $C \in E(M)$.

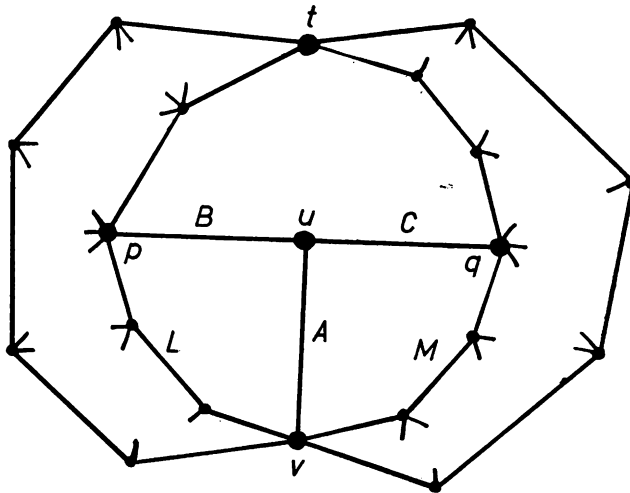


Fig. 5

It may happen that $p=t$. Then let L_1 be the subgraph of G defined by the edges of $E(L) - \{B\}$ and their incident vertices. It has an edge D incident with v , by (2.2), and no vertex of attachment other than t and v . Hence L_1 consists solely of the edge D and its two ends t and v . But then $\{A, B, D\}$ is a triangle and the theorem holds.

A similar argument applies if $q=t$. Accordingly we may assume that t is distinct from p and q . (Fig. 5.)

By another application of (2.2) $G'(B)$ is the union of two subgraphs Q and R with the following properties:

- (i) $E(Q) \cap E(R) = \emptyset$,
- (ii) Q and R have just three common vertices, r , u and p ,
- (iii) Each of r , u and p is incident with an edge of Q and an edge of R .

By arguments like those used for L and M we may assume that $A \in E(Q)$, $C \in E(R)$ and r is distinct from v and q . (Fig. 6.)

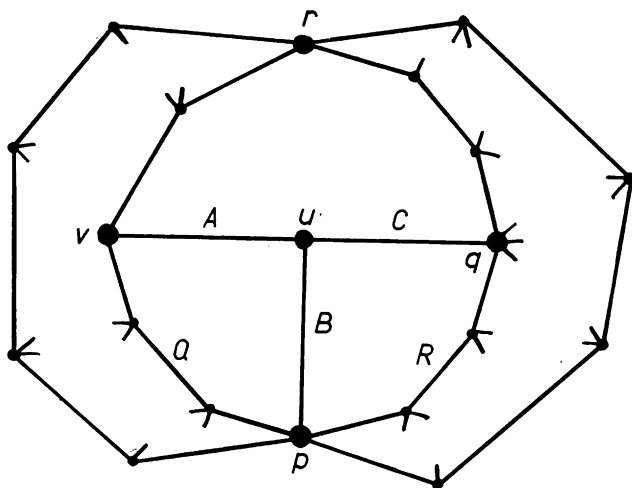


Fig. 6

Suppose $r \in V(L)$. Applying (3.2) we find that the only possible vertices of attachment of $M \cap R$ are u , r and t . But if r is such a vertex of attachment it is in $V(M)$ as well as $V(L)$, and is therefore t . Since $C \in E(M \cap R)$ we deduce that $M \cap R$ consists solely of the edge C with ends u and t . This is impossible since $q \neq t$.

We deduce that $r \in V(M) - \{t\}$. The only possible vertices of attachment of $L \cap R$ are t , u and p , by (3.2). But u is an isolated vertex of $L \cap R$. (val $(L \cap R, u) = 0$.) The graph J_1 obtained from $L \cap R$ by suppressing u has at most two vertices of attachment, t and p . On the other hand $E(J_1)$ is not null; it includes each edge of R incident with p . Accordingly J_1 consists of a single edge D with its ends p and t . This implies that $t \in V(R) - \{r\}$.

Now consider $L \cap Q$. By (3.2) its only possible vertices of attachment are u , v and p , since $r \in V(M) - \{t\}$ and $t \in V(R) - \{r\}$. But u is an isolated vertex of $L \cap Q$. The graph J_2 obtained from $L \cap Q$ by suppressing u has at most two vertices of attachment, v and p . On the other hand $E(J_2)$ is not null; it includes each edge of L incident with v . Hence J_2 consists of a single edge E with ends v and p . But then $\{A, B, E\}$ is a triangle of G .

4. THE MAIN THEOREM

(4.1) *Let G be a simple 3-connected graph with at least 4 vertices and in which every edge is essential. Then G is a wheel.*

Proof. Choose an edge A_0 of G with ends a and b_0 . By (3.3) and (3.5) A_0 belongs to a triangle $\{A_0, B_0, A_1\}$. We can suppose the ends of B_0 to be b_0 and b_1 , and those of A_1 to be a and b_1 .

By (3.4) B_0 belongs to a triad. We can adjust the notation so that b_1 is the centre of this triad. We can then write the triad as $\{B_0, A_1, B_1\}$, where B_1 is distinct from A_0 . Let the end of B_1 other than b_1 be b_2 . Since G is simple b_2 is distinct from a , b_0 , and b_1 .

By (3.5) B_1 belongs to a triangle, which can be written as $\{A_1, B_1, A_2\}$ or $\{B_0, B_1, A_2\}$. We can however adjust the notation, interchanging B_0 and A_1 , if necessary, so that the triangle is $\{A_1, B_1, A_2\}$. Then the ends of A_2 are a and b_2 , and A_2 is distinct from A_0 , A_1 , B_0 and B_1 . (Fig. 7.)

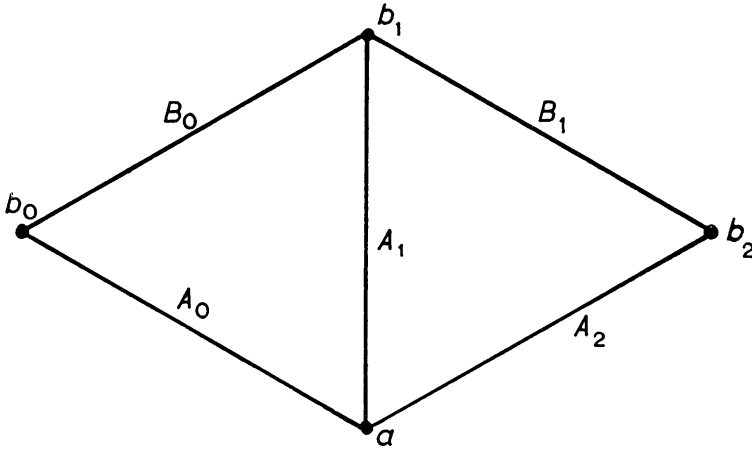


Fig. 7

Suppose a has valency 3 in G . Let H be the subgraph of G defined by the edges A_0, A_1, A_2, B_0, B_1 and their incident vertices. Then H has at most two vertices of attachment, b_0 and b_2 , since b_1 is the centre of the triad $\{B_0, A_1, B_1\}$. On the other hand G has a sixth edge incident with b_0 , by (2.3). Since G is simple and 3-connected it must be formed from H by the adjunction of a single edge B_2 with ends b_0 and b_2 . But then G is a wheel of order 3.

From now on we may assume that a is incident with at least four edges of G . We can assert that there is a subgraph H_j of G , for some $j \geq 2$, with the following specification and properties:

- (i) $E(H_j) = \{A_0, A_1, \dots, A_j, B_0, B_1, \dots, B_{j-1}\}$,
- (ii) $V(H_j) = \{a, b_0, b_1, \dots, b_j\}$,
- (iii) The ends of A_i are a and b_i , and those of B_i are b_i and b_{i+1} ,
- (iv) The $2j+1$ edges listed in (i) are all distinct, and the $j+1$ vertices listed in (ii) are all distinct,
- (v) Each of the vertices b_1, b_2, \dots, b_{j-1} has valency 3 in G , and the valency of a in G is at least four,
- (vi) j has the greatest value consistent with the above conditions.

By (3.4) A_j belongs to a triad. The centre of this triad is b_j , by (iii) and (v). We can write the triad as $\{B_{j-1}, A_j, B_j\}$, where B_j is distinct from all the edges listed in (i), by (iii) and (iv). Let the other end of B_j be b_{j+1} . This is distinct from a , since G is simple, and from all the vertices b_1, b_2, \dots, b_n , by (iii) and (v).

Suppose first that $b_{j+1} \neq b_0$. Then B_j belongs to a triangle, by (3.5). Accordingly G has an edge A_{j+1} with ends a and b_{j+1} . This edge is distinct from B_j and from all the edges listed in (i). Adjoining B_j , b_{j+1} and A_{j+1} to H_j we obtain a subgraph H_{j+1} of G satisfying all the conditions (i) to (v) with j replaced by $j+1$. This is contrary to (vi).

We deduce that $b_{j+1} = b_0$. Adjoining B_j to H_j we obtain a wheel W_{j+1} of order $j+1$. The only possible vertices of attachment of W_{j+1} are a and b_0 , by (iii) and (v). Hence $V(W_{j+1}) = V(G)$ by (3.1). Since G is simple, and a and b_0 are already joined by an edge A_0 of W_{j+1} , it follows that $G = W_{j+1}$.

5. THE CONSTRUCTION OF SIMPLE 3-CONNECTED GRAPHS

Two simple graphs G and H are *isomorphic* if there is a 1-1 mapping f of $V(G)$ and $V(H)$ such that $f(x)$ and $f(y)$ are joined by an edge of H if and only if x and y are joined by an edge of G ; $((x, y) \in V(G))$. In what follows we shall not regard isomorphic graphs as different.

Suppose we are given a complete list L_m of the (non-isomorphic) simple 3-connected graphs of m edges, (where $m \geq 3$). We consider the problem of determining the list L_{m+1} of all simple 3-connected graphs of $m+1$ edges.

Suppose $G \in L_{m+1}$. Since $m+1 \geq 4$ the graph G has at least 4 vertices. So by (4.1) either G is a wheel of order $\frac{1}{2}(m+1)$ or it has a non-essential edge A . The former case arises only when m is odd and > 3 . We note that any two wheels of the same order are isomorphic. In the second case either $G'(A) \in L_m$ or $G''(A) \in L_m$. We conclude that if G is not a wheel it can be derived from some $H \in L_m$ by one of the following operations:

(I) *Adjoining to H a new edge A whose ends are two distinct members of $V(H)$ not joined in H ,*

(II) *"Splitting" a vertex x of H incident with 4 or more edges, and adjoining A as an edge incident with the two resulting new vertices.*

The second operation is more precisely defined as follows: The edges incident with x are put into two disjoint classes U and V such that $\alpha(U) \geq 2 \leq \alpha(V)$. Then x is replaced by two distinct new vertices u and v and it is postulated that their incident edges are the members of U and V respectively. The incidence relations not involving x are left unchanged. Finally A is adjoined as a new edge with ends u and v . The graph G thus constructed evidently satisfies $G''(A) = H$ (to within an isomorphism). The necessity for the condition $\alpha(U) \geq 2 \leq \alpha(V)$ follows from (2.3) and the postulate that G is simple and 3-connected.

To complete the theory we need the following two propositions.

(5.1) *Let G be derived from $H \in L_m$ by operation (I). Then $G \in L_{m+1}$.*

Proof. Clearly G is simple. If it is not 3-connected let $\{P, Q\}$ be one of its 2-separators. We may assume $A \in E(P)$. Then $\{P'(A), Q\}$ is a 2-separator of H , contrary to hypothesis.

(5.2) *Let G be derived from $H \in L_m$ by operation (II). Then $G \in L_{m+1}$.*

Proof. Clearly G is simple. If it is not 3-connected let $\{P, Q\}$ be one of its 2-separators. We may assume $A \in E(P)$. Let P_0 and Q_0 be the subgraphs of H defined by the edges of $E(P) - \{A\}$ and $E(Q)$ respectively, with their incident vertices. We have $P_0 \cup Q_0 = H$, by (2.3).

If P_0 has a vertex not in $V(Q_0)$ it is clear that $\{P_0, Q_0\}$ is a 2-separator of H , contrary to hypothesis. We deduce that $V(P_0) \subseteq V(Q_0)$. This can happen only if A has one end u in $V(P) \cap V(Q)$ and the other v in $V(P) - V(Q)$, and then only if there is no other vertex in $V(P) - V(Q)$. But under these conditions it follows from the condition $\alpha(U) \geq 2 \leq \alpha(V)$ of operation (II) that v is joined by three edges of P to three distinct vertices of $V(P) \cap V(Q)$. This is contrary to the definition of P .

To construct L_{m+1} we first take note of the wheel of order $\frac{1}{2}(m+1)$ if m is odd and > 3 ; see (2.4). Then we apply operations (I) and (II) to each graph H of L_m . In each case we apply them in all possible ways not equivalent under the symmetry of H . The resulting graphs are simple and 3-connected, by (2.4), (5.1) and (5.2), and by (4.1) they include all the members of L_{m+1} . Striking out duplicates we obtain the required list L_{m+1} .

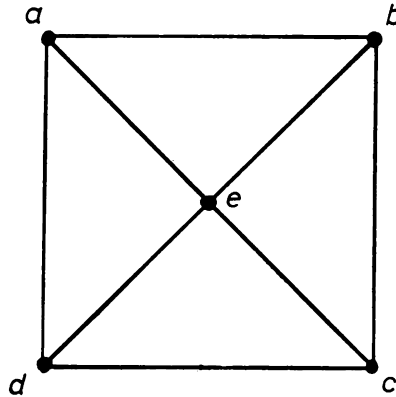


Fig. 8

Let us start this process with L_3 whose only member is the complete 3-graph. (The *complete n -graph* is defined as the simple graph of n vertices in which each pair of vertices is joined by an edge.) There is no way of applying (I) and (II) to this. Hence L_4 and L_5 have no members, and the only member of L_6 is the wheel of order 3, that is the complete 4-graph.

Neither (I) nor (II) can be applied to the complete 4-graph. Hence L_7 is null and the only member of L_8 is the wheel of order 4. (Fig. 8.) There is essentially only one way of applying operation (I) to this; we join a and c . Operation (II) can be applied only at e , but it can be applied there in two distinct ways. The resulting three graphs of L_9 are shown in Fig. 9.

The list L_{10} proves to have 4 members. These are given in Fig. 10.

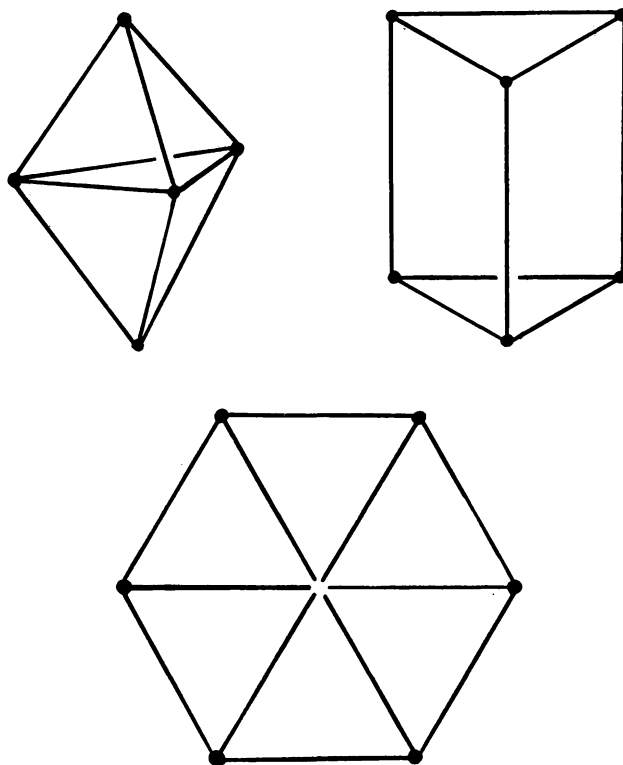


Fig. 9

6. PLANAR GRAPHS

It follows from the work of HASSLER WHITNEY [2, 3] that if a simple 3-connected graph G can be represented in the plane it can be so represented in essentially only one way. If there are at least three vertices the residual regions are simply connected domains bounded by polygons made up of edges of the graphs, and these polygons are uniquely determined as sets of edges by the structure of the graph. Let us call the subgraph of G defined by one of these polygonal sets of edges, with their incident vertices, a *face* of G . We exploit the uniqueness of the set of faces of G as follows.

Consider a simple 3-connected planar graph G derived from a simple 3-connected graph H by operation (I). If we represent G in the plane and delete the edge A we obtain a planar representation of H . We deduce

that H is planar and that G can be derived from it by joining two vertices of the same face which are not directly joined in H .

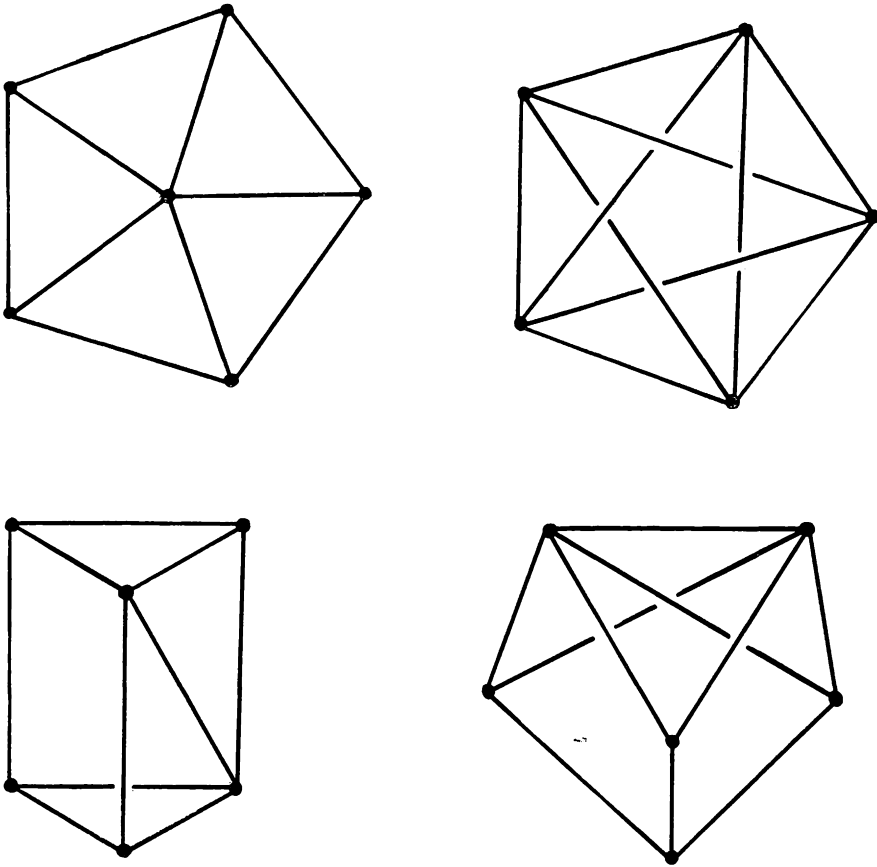


Fig. 10

Next suppose G is derived from a simple 3-connected graph H by operation (II). Let G^* be a dual graph of G [2] and let H_1 be the graph obtained from G^* by deleting the edge corresponding to A . It is readily verified that the dual graph of H_1 is isomorphic with H . If H has more than 3 vertices it follows that H_1 is simple and 3-connected. For a 2-separator of H_1 would determine a 2-separator of the dual graph defined by the corresponding partition of the set of edges. G^* is of course derived from H_1 by joining two vertices of the same face.

From the foregoing results we deduce:

(6.1) *Let G be a simple 3-connected planar graph with at least 4 vertices. Suppose further that G is not a wheel. Then either G or its dual graph can be derived from a simple 3-connected planar graph H , satisfying $\alpha_1(H) = \alpha_1(G) - 1$, by adjoining a new edge A whose ends are vertices of the same face of H and are not joined by an edge of H .*

Theorem (6.1) was communicated by the author to Dr. C. J. BOUWKAMP some years ago. He and his colleagues have made use of it in their work on the tabulation of the simple squared rectangles [1]. Perhaps it will also have applications in inductive proofs of theorems about 3-connected planar graphs.

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REFERENCES

1. BOUWKAMP, C. J., A. J. W. DUIJVESTIJN and P. MEDEMA. *Tables relating to simple squared rectangles of orders nine through fifteen*. Dept. of Mathematics and Mechanics, Technische Hogeschool, Eindhoven. (1960).
2. HASSLER WHITNEY, *Non-separable and planar graphs*, Trans. Amer. Math. Soc., **34**, 339–362 (1932).
3. ———, *2-Isomorphic graphs*, Amer. J. Math., **55**, 245–254 (1933).